

HILBERT'S MACHINE AND ω -ORDER

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ABSTRACT. Hilbert's machine is a supertask machine inspired by Hilbert's Hotel whose functioning leads to a contradictory result involving ω -ordering and then the actual infinity.

1. INTRODUCTION

In the next discussion we will make use of a supermachine inspired by the emblematic Hilbert's Hotel. But before beginning, let us relate some of the prodigious (and suspicious) abilities of the illustrious Hotel. Its director, for instance, has discovered a fantastic way of getting rich: it demands one euro to the guest of the room 1; this guest recover his euro by demanding one euro to the guest of the room 2; the guest of the room 2 recover his euro by demanding one euro to the guest of the room 3, and so on. Finally all guests recover his euro, and then our crafty director demands a second euro to the guest of the room 1 which recover again his euro by demanding one euro to the guest of the room 2, which recover again his euro by demanding one euro to the guest of the room 3, and so on and on. Thousands of euros coming from the (infinetist) nothingness to the pocket of our fortunate director.

Hilbert's Hotel is even capable of violating the laws of thermodynamics making it possible the functioning of a perpetuum mobile: in fact we would only have to power the appropriate machine with the calories obtained from the successive rooms of the prodigious hotel in the same way its director got his euros. It is in fact unbelievable that infinitists justify all those pathologies, and many other, in behalf of the *peculiarities* of the actual infinity. It is unbelievable that they prefer to assume the pathological behaviour of the world before questioning the consistency of the pathogene. But in the next discussion we will come to a contradiction that cannot be easily subsumed in the picturesque nature of the actual infinity, a contradiction from which it is impossible to escape.

2. HILBERT'S MACHINE

In the following conceptual discussion we will make use of a theoretical device that will be referred to as *Hilbert's machine*, composed of the following elements:

- (1) An infinite magnetic wire which is divided into two infinite parts, the left and the right side:
 - (a) The right side is divided into an ω -ordered sequence of adjacent magnetic sections $\langle s_i \rangle_{i \in \mathbb{N}}$ which are indexed from left to right as s_1, s_2, s_3, \dots . They will be referred to as right sections.
 - (b) The left side is also divided into an ω -ordered sequence of adjacent sections $\langle s'_i \rangle_{i \in \mathbb{N}}$ indexed now from right to left as \dots, s'_3, s'_2, s'_1 ; being s'_1 adjacent to s_1 . They will be referred to as left sections.
- (2) An ω -ordered sequence of magnetic sliding beads $\langle b_i \rangle_{i \in \mathbb{N}}$ which are inserted in the magnetic wire as the beads of an abacus, being each bead b_i initially placed on the right section s_i .
- (3) A magnetic multidisplacement mechanism which moves simultaneously each bead exactly one section to the left, so that the bead placed on $s_{k, k > 1}$ is placed on s_{k-1} , the one placed on s_1 is placed on s'_1 , and if one were placed on s'_k it would be placed on s'_{k+1} . This simultaneous displacement of all beads $\langle b_i \rangle_{i \in \mathbb{N}}$ one section to the left will be termed *magnetic multidisplacement*, or simply *multidisplacement*. Multidisplacements are the *only actions* Hilbert's machine can perform.

Let us now consider the following definition: we will say that a bead b_i is removed from the wire if, and only if, it is placed out of the wire as a consequence of a multidisplacement. Although the impossibility of being removed from the infinite magnetic wire¹ would facilitate our discussion, we will assume that removal is, nevertheless, possible. Consequently, we will impose the following restriction to the functioning of Hilbert's machine: the machine will perform a multidisplacement if, and only if, the multidisplacement does not remove any bead from the wire nor alters the original ω -order of the beads b_1, b_2, b_3, \dots (Hilbert's restriction).

Assume now that Hilbert's machine performs a magnetic multidisplacement m_i at each one of the countably many instants t_i of any ω -ordered sequence of instants $\langle t_i \rangle_{i \in \mathbb{N}}$ defined within any finite interval of time (t_a, t_b) , for instance the classical

¹The last left magnetic section of the wire does not exist.

one defined by:

$$t_i = t_a + (t_b - t_a) \sum_{k=1}^i \frac{1}{2^k}, \quad \forall i \in \mathbb{N} \quad (1)$$

whose limit is t_b . In these conditions, at t_b our machine will have completed the ω -ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$, i.e. a supertask. As is usual in supertask theory² we will also assume that multidisplacements are instantaneous. Although it is irrelevant for our conceptual discussion, we could also assume that each multidisplacement lasts a finite amount of time, for instance each m_i could take a time $1/(2^{i+1})$. It seems appropriate at this point to emphasize the conceptual nature of the discussion that follows. We are not interested here in discussing the problems derived from the actual performance of supertasks in our physical universe, as would be the case of the length of the wire or the relativistic restrictions on the speed of the magnetic multidisplacements and the like³. We will assume, therefore, that Hilbert's machine works in a conceptual universe in which no physical restriction limits its functioning. Our only objective here is to examine the consistency of ω -ordering.

3. PERFORMING THE SUPERTASK

Consider the ω -ordered sequence of instants $\langle t_i \rangle_{i \in \mathbb{N}}$ defined according to (1), and a Hilbert's machine in the following initial conditions:

- (1) At t_a the machine is at rest.
- (2) At t_a each bead b_i is on the right section s_i .
- (3) At t_a each left section s'_i is empty.

Assume that, if Hilbert's restriction allows it, this machine performs exactly one magnetic multidisplacement m_i at each one of the countably many instants t_i of $\langle t_i \rangle_{i \in \mathbb{N}}$, and only at them, being those successive multidisplacements the only performed actions.

An ω -ordered sequence is one in which there exists a first element and each element has an immediate successor. Consequently no last element exists. Thus, ω -ordered sequences are both complete (as the actual infinity requires) and uncompletable (in the sense that no last element completes them). The objective of the following discussion is just to analyze the consequence of completing the uncompletable ω -ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$. We begin by proving the following basic proposition which is directly derived from assuming the existence of ω -ordered sequences as complete totalities.

²See [5], [11], [12], [14], [15], [18], [10], etc.

³[8], [9], [14], [15], [16], [10], [17], [13], [1], [2], [19], [4], [3], etc.

Proposition 1. ω -order makes it possible that all multidisplacements m_i of the ω -ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$ observe Hilbert's restriction.

Proof. It is evident the first multidisplacement m_1 observes Hilbert's restriction. In fact, according to ω -order each right section $s_{i, i>1}$ has an immediate predecessor to the left and each left section has an immediate successor to the left. On the other hand, s_1 has its own immediate predecessor to the left: s'_1 . Thus, b_1 can be moved to s'_1 and each $b_{i, i>1}$ to s_{i-1} . Consequently m_1 does not remove any bead from the wire nor alter the original ω -order of the beads since each bead b_i remain succeeded by its original immediate successor b_{i+1} because all of them are simultaneously moved one section the left. Assume the first n multidisplacements observe Hilbert's restriction. Since each multidisplacement moves each bead exactly one section to the left, after performing these first n multidisplacements, b_1 will have been placed on s'_n , $b_{i, 1 < i \leq n}$ on s'_{n-i+1} and $b_{i, i > n}$ on s_{i-n} . All these sections have an immediate predecessor (or successor) to the left so that all beads can be simultaneously moved one section to the left without removing any bead from the wire nor altering the initial ω -order of the beads because each beads b_i remains succeeded by its original immediate successor b_{i+1} for the same reasons above. In consequence, multidisplacement m_{n+1} also observes Hilbert's restriction. We have just proved that m_1 observes Hilbert's restriction, and that if the first n multidisplacements observe Hilbert's restriction, then m_{n+1} also observes Hilbert's restriction. Therefore, every multidisplacement m_i observes Hilbert's restriction. \square

It is now possible to prove the following two contradictory results:

Proposition 2. At t_b the ω -ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$ has been completed.

Proof. According to Proposition 1 all multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$ observe Hilbert restriction. Consequently all of them can be performed by Hilbert's machine without removing any bead from the wire nor altering the initial ω -order of the magnetic beads. We will prove now that at t_b all multidisplacements have been carried out. For this, consider the one to one correspondence f between $\langle t_i \rangle_{i \in \mathbb{N}}$ and $\langle m_i \rangle_{i \in \mathbb{N}}$ defined by:

$$f(t_i) = m_i, \quad \forall i \in \mathbb{N} \quad (2)$$

Being t_b the limit of the ω -ordered sequence $\langle t_i \rangle_{i \in \mathbb{N}}$, and taking into account that each multidisplacement m_i takes place at the precise instant t_i , the one to one correspondence f together with the assumed completeness of all ω -ordered sequences, prove that at t_b all multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$ have been carried out.

At t_b , therefore, the ω -ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$ has been completed. \square

Proposition 3. *At t_b the ω -ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$ has not been completed.*

Proof. According to Hilbert restriction and Proposition 1 no bead is removed from the wire and the initial ω -order of $\langle b_i \rangle_{i \in \mathbb{N}}$ is preserved. Let therefore b_n be any bead. Since it has not been removed from the wire, at t_b it must of necessity be in on one of its magnetic sections. Assume it is on s_k . Since each multidisplacement moves b_n a section to the left, only a finite number $n - k$ of multidisplacements will have been carried out to move b_n from its initial section s_n to s_k . Assume now b_n is on s'_h at t_b . In this case, and for the same reason above, only a finite number $n + h$ of multidisplacements will have been performed. We can therefore conclude that at t_b only a finite number of multidisplacements can have been performed. So, at t_b the ω -ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$ has not been completed. \square

4. CONSEQUENCES

We have just proved that at t_b the ω -ordered sequence of multidisplacements $\langle m_i \rangle_{i \in \mathbb{N}}$ has and has not been completed⁴. Obviously, Hilbert's machine is a conceptual device whose theoretical existence and functioning is only possible under the assumption of ω -order, that legitimates the *completeness* of the ω -ordered sequences $\langle s_i \rangle_{i \in \mathbb{N}}$, $\langle s'_i \rangle_{i \in \mathbb{N}}$, $\langle b_i \rangle_{i \in \mathbb{N}}$, $\langle m_i \rangle_{i \in \mathbb{N}}$ and $\langle t_i \rangle_{i \in \mathbb{N}}$. Furthermore, the contradictory Propositions 2 and 3 are formal consequences of Proposition 1, which in turn is a formal consequence of ω -order. It is, therefore, ω -order the cause of the contradiction between Propositions 2 and 3.

We will come to the same conclusion on the inconsistency of ω -order by comparing the functioning of the above infinite Hilbert's machine (symbolically H_ω) with the functioning of any finite Hilbert machine with a finite number n of both right and left sections (symbolically H_n); being, as in the case of H_ω , a sequence of n magnetic beads initially placed in the right side of the wire, each bead b_i on the section s_i . In effect, it is immediate to prove that, according to Hilbert's restriction, H_n can only perform n multidisplacements because the $(n + 1)$ -th multidisplacement would remove from the wire the bead b_1 initially placed on the first right section s_1 and placed on the last left section s'_n by multidisplacement m_n . Thus m_{n+1} does not observe Hilbert restriction and the machine halts before

⁴Although we will not do it here, it is possible to derive other contradictory results from the functioning of Hilbert's machine

performing m_{n+1} . H_n halts with each left section s'_i occupied by the bead b_{n-i+1} and all right sections empty, and this is all. No contradiction is derived from the functioning of H_n . Thus for any natural number n , H_n is consistent. Only infinite Hilbert's machine H_ω is inconsistent. Consequently, and taking into account that ω -order is the only difference between H_ω and $H_n, \forall n \in \mathbb{N}$, only ω -order can be the cause of the inconsistency of H_ω .

What the above contradiction proves, therefore, is not that a particular supertask is inconsistent. What it proves is the inconsistency of ω -order itself. Perhaps we should not be surprised by this conclusion. After all, an ω -ordered sequence is one which is both complete (as the actual infinity requires) and uncompletable (there is not a last element that completes it). On the other hand, and as Cantor proved [6], [7], ω -order is an inevitable consequence of assuming the existence of denumerable complete totalities. An existence axiomatically stated in our days by the Axiom of Infinity, in all axiomatic set theories including ZFC and BNG [21], [20]. It is therefore that axiom the ultimate cause of the contradiction between Propositions 2 and 3.

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